

SOME TOPICS ON RICCI SOLITONS AND SELF-SIMILAR SOLUTIONS TO MEAN CURVATURE FLOW

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ABSTRACT. In this survey article, we discuss some topics on self-similar solutions to the Ricci flow and the mean curvature flow. Self-similar solutions to the Ricci flow is known as Ricci solitons. In the first part of this paper we discuss a lower diameter bound for compact manifolds with shrinking Ricci solitons. Such a bound can be obtained from an eigenvalue estimate for a twisted Laplacian, called the Witten-Laplacian. In the second part we discuss self-similar solutions to the mean curvature flow on cone manifolds. Many results have been obtained for solutions in \mathbb{R}^n or \mathbb{C}^n . We see that many of them extend to cone manifolds, and in particular results on \mathbb{C}^n for special Lagrangians and self-shrinkers can be extended to toric Calabi-Yau cones. We also see that a similar lower diameter bound can be obtained for self-shrinkers to the mean curvature flow as in the case of shrinking Ricci solitons.

1. INTRODUCTION

In this survey paper we discuss self-similar solutions to the two major geometric flows, the Ricci flow and the mean curvature flow. The self-similar solutions to the Ricci flow are called the Ricci solitons. The self-similar solutions appear as rescaling limits of the singularities of the corresponding flows, see [20], [34], [5] for the Ricci flow, and [22] for the mean curvature flow. The two kinds of self-similar solutions have common aspects. A typical such aspect is that there is a common eigenvalue -2λ for the twisted Laplacian $\Delta_f = \Delta - \nabla f \cdot$, that is

$$\Delta_f u = \Delta u - g^{ij} \nabla_i f \nabla_j u$$

where

$$\text{Ric}(g) - \gamma g + \nabla \nabla f = 0$$

in the case of Ricci flow (see Step 1 in the proof of Theorem 2.4), and where

$$\vec{H} = -\lambda x^\perp$$

and

$$f = 2\lambda \left(\frac{|x|^2}{4} - \frac{n}{4\lambda} \right)$$

in the case of mean curvature flow $x : M^n \rightarrow \mathbb{R}^{n+p}$ (see Theorem 6.1). Combining this with an eigenvalue estimate for Δ_f given in Step 2 in the proof of Theorem 2.4 we obtain lower diameter bounds both for the Ricci soliton (Theorem 2.4) and the immersed submanifold of the self-similar solution to the mean curvature flow (Theorem 6.2). These two are treated in section 2 and section 6.

Date: June 4, 2012 .

2000 Mathematics Subject Classification. Primary 53C55, Secondary 53C21, 55N91 .

Key words and phrases. Ricci flow, Ricci soliton, mean curvature flow, self-similar solution.

In section 3 we study the mean curvature flow and its self-similar solutions on Riemannian cone manifolds. The self-similar solutions of the mean curvature flow have been studied for immersions into \mathbb{R}^{n+p} because we need to have position vectors and their orthogonal projection to define self-similar solutions. The idea of [15], on which this section is based, is that there are natural position vectors and orthogonal projections for immersions into Riemannian cone manifolds. We propose to study the self-similar solutions of the mean curvature flow into Riemannian cone manifolds because many earlier works extend to the cone situation. As a typical such result we see that the self-similar solutions are obtained as the limit of parabolic rescalings of the type I_c singularity, extending an earlier work of Huisken [22].

After a brief introduction to toric Sasaki-Einstein manifolds in section 4, we give in section 5 a construction of special Lagrangian submanifolds in toric Calabi-Yau cones, extending an earlier example in \mathbb{C}^n by Harvey-Lawson [21]. This section is based on [15]. Yamamoto [36] has further constructions of compact Lagrangian self-shrinkers.

In section 7 eternal solutions to the Ricci flow are constructed on certain line bundles over toric Fano manifolds. This section is based on [19].

Acknowledgement. This paper is partly based on the author's notes of the talk delivered at the Modern Mathematics Seminar Series of Mathematical Sciences Center of Tsinghua University in September 2011, and the later discussions with H.-Z. Li and X.-D. Li. The author would like to express his gratitude to Professor S.-T. Yau and Prof. Y.-S. Poon for the invitation to the MSC and hospitality.

2. RICCI SOLITONS

A complete Riemannian metric g on a smooth manifold M is called a *Ricci soliton* if there is a $\gamma \in \mathbb{R}$ and a vector field X such that

$$2 \operatorname{Ric}(g) - 2\gamma g + \mathcal{L}_X g = 0.$$

If $X = 0$ then $\operatorname{Ric}(g) = \gamma g$, i.e. g is Einstein. In this case we say g is *trivial*. We say that a nontrivial soliton is *expanding*, *steady* or *shrinking* according as $\gamma < 0$, $\gamma = 0$ or $\gamma > 0$.

If $X = \operatorname{grad} f$ for some $f \in C^\infty(M)$ then

$$\operatorname{Ric}(g) - \gamma g + \nabla \nabla f = 0.$$

In this case g is called a *gradient soliton*.

Given a Ricci soliton, let Y_t be the time dependent vector field

$$Y_t := -\frac{1}{2\gamma t} X$$

and let φ_t be the flow generated by Y_t . If we set

$$g(t) = -2\gamma t \varphi_t^* g$$

then $g(t)$ satisfies the Ricci flow equation

$$\frac{\partial g(t)}{\partial t} = -2 \operatorname{Ric}(g(t)).$$

A Ricci soliton is a self-similar solution to the Ricci flow equation since it is obtained as a rescaling limit of a singularity ([20], [34], [5]).

Theorem 2.1 (Perelman [32], see also [11]). *Any soliton on a compact manifold is a gradient soliton.*

Theorem 2.2 (Hamilton [20], Ivey [23], see also [4]). *Any nontrivial gradient Ricci soliton on a compact manifold is shrinking with $\dim M \geq 4$.*

All known examples of compact Ricci solitons are Kähler Ricci solitons:

- 1) Koiso [26] and Cao [3] : \mathbb{P}^1 -bundles over products of \mathbb{P}^n 's;
- 2) Wang-Zhu [35]: toric Fano manifolds;
- 3) Podesta-Spiro [33]: homogeneous toric bundles.

Kähler-Ricci solitons are Kähler-Einstein metrics exactly when the invariant in [13] vanishes.

For the constructions of solitons on non-compact manifolds, see for example [10] and [19].

Problem 2.3. *Does there exist a compact non-Kähler nontrivial gradient soliton ?*

The first topic in this survey is about the following lower diameter bound for compact gradient shrinking Ricci solitons.

Theorem 2.4 (Futaki-Li-Li [16]). *Let M^n be a compact manifold with $\dim M = n \geq 4$. If g is a nontrivial gradient shrinking soliton on M with*

$$2 \operatorname{Ric}(g) - 2\gamma g + \mathcal{L}_X g = 0$$

then

$$d_g \geq \frac{2(\sqrt{2}-1)}{\sqrt{\gamma}}\pi$$

where d_g is the diameter of (M, g) .

As an immediate corollary we have

Corollary 2.5. *If a compact gradient shrinking soliton has*

$$(1) \quad d_g < \frac{2(\sqrt{2}-1)}{\sqrt{\gamma}}\pi$$

then g is a trivial soliton (i.e. Einstein).

It is interesting to compare Corollary 2.5 with the following theorem of Meyers. When g is Einstein with $\operatorname{Ric} = \gamma g$, i.e. $X = 0$, and with $\gamma > 0$ then Meyers' theorem says

$$d_g \leq \sqrt{\frac{n-1}{\gamma}}\pi.$$

Problem 2.6. *Does there exist an example of Einstein manifold satisfying (1)?*

The proof of Theorem 2.4 is given as follows.

Proof of Theorem 2.4.

Step 1 (cf. [18]) :

Suppose we have a gradient Ricci soliton

$$\operatorname{Ric}(g) - \gamma g + \nabla \nabla f = 0.$$

Define Δ_f by

$$\Delta_f u = \Delta u - \nabla f \cdot \nabla u (= g^{ij} \nabla_i \nabla_j u - \nabla^i f \nabla_i u).$$

We normalize f so that

$$\int_M f e^{-f} dV_g = 0.$$

Then -2γ is an eigenvalue of Δ_f . In fact

$$(2) \quad \Delta_f f + 2\gamma f = 0.$$

Step 2 ([16]) :

If $\Delta_f u + \lambda u = 0$ for some nonzero $u \in C^\infty(M)$. Then

$$(3) \quad \lambda \geq \sup_{s \in (0,1)} \{4s(1-s) \frac{\pi^2}{d^2} + s\gamma\}$$

This Step 2 is the essential part, and its proof is explained later.

Step 3:

By Step 1 and Step 2 we have for any $s \in (0, 1)$

$$2\gamma \geq 4s(1-s) \frac{\pi^2}{d^2} + s\gamma,$$

and hence

$$(4) \quad \gamma \geq \frac{4s(1-s)}{2-s} \frac{\pi^2}{d^2}.$$

The right hand side of (4) takes maximum

$$(5) \quad \frac{4s(1-s)}{2-s} \leq 12 - 8\sqrt{2}$$

at $s = 2 - \sqrt{2} \in (0, 1)$. From (4) and (5) we get

$$\begin{aligned} d_g &\geq \sqrt{\frac{12 - 8\sqrt{2}}{\gamma}} \pi \\ &= \frac{2(\sqrt{2} - 1)}{\sqrt{\gamma}} \pi \end{aligned}$$

This completes the proof of Theorem 2.4. \square

Now we turn to the proof of Step 2 in the proof above. We apply the following result.

Theorem 2.7 ([6], [7], [2], [1]). *Let (M, g) be a compact Riemannian manifold, and ϕ be a C^2 function on M . Suppose that*

$$\text{Ric}(g) + \nabla \nabla \phi \geq Kg$$

for some $K \in \mathbb{R}$. Then the first nonzero eigenvalue λ_1 for the Witten-Laplacian

$$\Delta_\phi = \Delta - \nabla^i \phi \cdot \nabla_i$$

satisfies

$$\lambda_1 \geq \lambda_1(L)$$

where $\lambda_1(L)$ is the first nonzero Neumann eigenvalue of

$$L = \frac{d^2}{dx^2} - Kx \frac{d}{dx}$$

on $(-d/2, d/2)$.

If this theorem is granted, the proof of Step 2 is obtained from the following.

Theorem 2.8 ([16]). *Under the notations as above we have*

$$\lambda_1(L) \geq \sup_{s \in (0,1)} \{4s(1-s)\frac{\pi^2}{d^2} + sK\}.$$

Proof. We set $\lambda := \lambda_1(L)$ for short. Let v be the first Neumann eigenfunction for L . If we put $D = \frac{d}{2}$ and $f = v'$ then f is the first Dirichlet eigenfunction for L with

$$(6) \quad f'' - Kxf' = -(\lambda - K)f,$$

and

$$f(-D) = f(D) = 0.$$

Since the first Dirichlet eigenfunction does not change sign we may assume $f > 0$ on $(-D, D)$. Take any $a > 1$. Then by (6) we have

$$(7) \quad \int_{-D}^D f^{a-1}(x) f''(x) dx = -(\lambda - K) \int_{-D}^D f^a(x) dx + \int_{-D}^D Kx f^{a-1}(x) f'(x) dx.$$

By integration by parts and $f^{a-1}(\pm D) = 0$ we have

$$(8) \quad \int_{-D}^D Kx f^{a-1}(x) f'(x) dx = -\frac{K}{a} \int_{-D}^D f^a(x) dx.$$

If we put $u = f^{\frac{a}{2}}$ then using (7) and (8) we get

$$\frac{4(a-1)}{a^2} \int_{-D}^D |u'|^2 dx = (\lambda - K(1 - \frac{1}{a})) \int_{-D}^D u^2 dx.$$

If we put $s = 1 - 1/a$ this is equivalent to

$$4s(1-s) \int_{-D}^D |u'|^2 dx = (\lambda - Ks) \int_{-D}^D u^2 dx.$$

Thus we obtain

$$\begin{aligned} \frac{\lambda - Ks}{4s(1-s)} &= \frac{\int_{-D}^D |u'|^2 dx}{\int_{-D}^D u^2 dx} \\ &\geq \frac{\pi^2}{4D^2} = \frac{\pi^2}{d^2} \end{aligned}$$

where the equality holds for $u = \sin(\frac{\pi}{2D}x + \frac{\pi}{2})$. It follows that, for any $s \in (0, 1)$ we have

$$\lambda \geq 4s(1-s)\frac{\pi^2}{d^2} + Ks.$$

This completes the proof of Theorem 2.8. \square

Remark 2.9. *Theorem 2.4 improves earlier results in [18] and [1]. In fact those results follow from weaker inequalities than (3). A principle behind those estimates is the “Bakry-Emery principle” [30] : If you have to replace the volume form dV_g by $e^{-f} dV_g$ for some reason, then replace Δ by*

$$\Delta_f := \Delta - \nabla \nabla f.$$

and replace Ric by

$$\text{Ric}_f = \text{Ric} + \nabla \nabla f.$$

Then a theorem for $(dV_g, \Delta, \text{Ric})$ extends to $(e^{-f}dV_g, \Delta_f, \text{Ric}_f)$.

3. THE MEAN CURVATURE FLOW AND SELF-SIMILAR SOLUTIONS ON RIEMANNIAN CONE MANIFOLDS.

Definition 3.1. A Riemannian cone manifold $(C(N), \bar{g})$ over (N, g) consists of a smooth manifold $C(N)$ diffeomorphic to $N \times \mathbb{R}_+$ and a Riemannian metric

$$\bar{g} = dr^2 + r^2g$$

on $C(N)$ where r is the standard coordinate on \mathbb{R}_+ .

Definition 3.2. Let $F : M \rightarrow (C(N), \bar{g})$ be an immersion. We call

$$\vec{F}(p) = r(F(p)) \frac{\partial}{\partial r} \in T_{F(p)}C(N)$$

the position vector of F at $p \in M$.

Example 3.3. If N is $(n-1)$ -dimensional standard sphere S^{n-1} we have $C(N) = \mathbb{R}^n - \{\mathbf{o}\}$. For an immersion $F : M \rightarrow \mathbb{R}^n - \{\mathbf{o}\}$ the position vector in the sense of Definition 3.2 is

$$\vec{F}(p) = r \frac{\partial}{\partial r} \in T_{F(p)}\mathbb{R}^n - \{\mathbf{o}\}.$$

But when $T_{F(p)}\mathbb{R}^n - \{\mathbf{o}\}$ is identified with \mathbb{R}^n , $\vec{F}(p)$ is identified with $F(p) \in \mathbb{R}^n$ which is exactly the position vector in the usual sense.

An immersion $F : M \rightarrow (C(N), \bar{g})$ is called a *self-similar solution* to the mean curvature flow if

$$H = \lambda \vec{F}^\perp$$

where H is the mean curvature vector at $F(p)$, and \perp denotes the orthogonal projection to the normal bundle. There are many works for self-similar solutions

$$F : M \rightarrow \mathbb{R}^n,$$

e.g. Huisken [22], Joyce-Lee-Tsui [24]. We expect that they are extended to

$$F : M \rightarrow C(N).$$

As a typical such example we start with the following result which extends a result of Huisken [22] in the case when $N = S^{n-1}$ and $C(N) = \mathbb{R}^n - \{\mathbf{o}\}$.

Theorem 3.4 (Huisken). *Let M be compact manifold and (N, g) a compact Riemannian manifold. Let $F : M \times [0, T) \rightarrow C(N)$ be a mean curvature flow with*

$$\frac{\partial F}{\partial t}(p, t) = H_t(p)$$

where H_t is the mean curvature vector of $F_t(M) := F(M, t)$. Then the maximal time T of existence of the flow is finite.

The following also extends a result of Huisken for $N = S^{n-1}$ and $C(N) = \mathbb{R}^n - \{\mathbf{o}\}$.

Theorem 3.5 (Monotonicity formula, due to Huisken). *Let $\rho_T : \mathbb{R} \times (-\infty, T) \rightarrow \mathbb{R}$ be the backward heat kernel*

$$\rho_T(y, t) = \frac{1}{(4\pi(T-t))^{n/2}} \exp\left(-\frac{y^2}{4(T-t)}\right).$$

Let $F : M \times [0, T) \rightarrow C(N)$ be the mean curvature flow. Then

$$\begin{aligned} & \frac{d}{dt} \int_M \rho_T(r(F_t(p)), t) dV_{g_t} \\ &= - \int_M \rho_T(r(F_t(p)), t) \left| \frac{\vec{F}_t^\perp(p)}{2(T-t)} + H_t(p) \right|_{\bar{g}}^2 dV_{g_t}. \end{aligned}$$

Definition 3.6 (Parabolic rescaling of scale λ). *Given a smooth map $F : M \times [0, T) \rightarrow C(N)$, the parabolic rescaling $F^\lambda : M \times [-\lambda^2 T, 0) \rightarrow C(N)$ of scale λ is defined by*

$$F^\lambda(p, s) = (\pi_N(F(p, T + \frac{s}{\lambda^2})), \lambda r(F(p, T + \frac{s}{\lambda^2})))$$

where $\pi_N : C(N) = N \times \mathbb{R} \rightarrow N$ is the standard projection.

Definition 3.7. *We say that a mean curvature flow F develops a type I singularity as $t \rightarrow T$ if there is a constant $C > 0$ such that*

$$\sup_M |\Pi_t|^2 \leq \frac{C}{T-t}$$

where Π_t is the second fundamental form of $F_t(M)$.

Huisken [22] has shown that if a mean curvature flow $F : M \times [0, T) \rightarrow \mathbb{R}^n$ develops a singularity of type I as $t \rightarrow T$ then for any increasing sequence $\{\lambda_i\}_{i=1}^\infty$ with $\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$, $\{F^{\lambda_i}\}$ subconverges to a self-similar solution $F^\infty : M_\infty \rightarrow \mathbb{R}^n$. To get a similar result in the case of general cone $C(N)$ we need the following definition.

Definition 3.8. *A mean curvature flow $F : M \times [0, T) \rightarrow C(N)$ develops a type I_c singularities $t \rightarrow T$ if*

- (a) F develops a type I singularity as $t \rightarrow T$;
- (b) $r(F_t(p)) \rightarrow 0$ for some $p \in M$ as $t \rightarrow T$;
- (c) For some positive constants K_1 and K_2 we have $K_1(T-t) \leq \min_M r^2(F_t) \leq K_2(T-t)$ for all $t \in [0, T)$.

The following is a cone version of Huisken's result that the limit of parabolic rescalings of a type I singularity yields a self-similar solution.

Theorem 3.9 ([15]). *Let M be an m -dimensional compact manifold and $C(N)$ the Riemannian cone manifold over an n -dimensional Riemannian manifold (N, g) . Let $F : M \times [0, T) \rightarrow C(N)$ be a mean curvature flow, and assume that F develops a type I_c singularity at T . Then, for any increasing sequence $\{\lambda_i\}_{i=1}^\infty$ of the scales of parabolic rescaling such that $\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$, there exist a subsequence $\{\lambda_{i_k}\}_{k=1}^\infty$ and a sequence $t_{i_k} \rightarrow T$ such that the sequence of rescaled mean curvature flow $\{F_{s_{i_k}}^{\lambda_{i_k}}\}_{k=1}^\infty$ with $s_{i_k} = \lambda_{i_k}^2(t_{i_k} - T)$ converges to a self-similar solution $F^\infty : M_\infty \rightarrow C(N)$ to the mean curvature flow.*

4. SASAKI MANIFOLDS

In this section we briefly review Sasakian geometry. See for example [14] for more details. A Riemannian manifold (N, g) with $\dim N = 2m + 1$ is called a Sasaki manifold if the cone manifold $(C(N), \bar{g})$ with $C(N) = N \times \mathbb{R}_+$, $\bar{g} = dr^2 + r^2 g$, is Kähler. Thus we have $\dim_{\mathbb{C}} C(N) = m + 1$. We often identify N with the submanifold $\{r = 1\}$ in $C(N)$.

The vector field $\xi = J(r \frac{\partial}{\partial r})$ is called the Reeb vector field. Then $\frac{1}{2}(\xi - iJ\xi)$ generates a holomorphic flow on $C(N)$. The local leaf spaces of $\frac{1}{2}(\xi - iJ\xi)$ on $C(N)$ is identified with the local leaf spaces of ξ on N . Here ξ preserves $N \cong \{r = 1\}$ and is considered as a vector field on N . Thus Reeb flow \mathcal{F}_ξ on N has a transverse holomorphic structure.

The dual 1-form η of ξ is a contact form on N . Thus $d\eta$ is non-degenerate on $\text{Ker } \eta \cong \nu(\mathcal{F}_\xi)$ and $d\eta/2$ gives \mathcal{F}_ξ a transverse Kähler structure. The contact form η can be lifted to a 1-form $\tilde{\eta}$ on $C(N)$, and $d\tilde{\eta} = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log r^2$ restricts to the transverse Kähler form on N . On the other hand, $\frac{\sqrt{-1}}{2} \partial \bar{\partial} r^2$ is the Kähler form on $C(N)$. A typical Example is the triple

$$(C(N), N, \text{leaf space}) = (\mathbb{C}^{m+1} - \{0\}, S^{2m+1}, \mathbb{CP}^m).$$

Simple curvature computations show that the following proposition.

Proposition 4.1. *The following three conditions are equivalent:*

- (a) $C(N)$ is Ricci-flat Kähler;
- (b) N is Einstein;
- (c) the local leaf spaces are positive Kähler-Einstein.

Again the triple $(\mathbb{C}^{m+1} - \{0\}, S^{2m+1}, \mathbb{CP}^m)$ is a typical example of Proposition 4.1.

Example 4.2. *Let M be a compact Kähler manifold and ω a Kähler form such that $[\omega]$ is an integral class. Let $p : L \rightarrow M$ be a holomorphic line bundle with $c_1(L) = -[\omega]$. We take an Hermitian metric h such that $i\partial\bar{\partial} \log h = \omega$ and put r to be the distance from the zero section along the fiber with respect to h . Let $N \subset L$ be the unit circle bundle (i.e. associated $U(1)$ -bundle). Then N is a Sasaki manifold, and $C(N) = L - \{\text{zero section}\}$ where Reeb field generates the S^1 -action. In this case η is the connection form of $L \rightarrow M$.*

Let N be a general Sasaki manifold. A smooth differential form α on N is said to be basic if

$$i(\xi)\alpha = 0 \quad \text{and} \quad \mathcal{L}_\xi \alpha = 0.$$

Denote by $\Omega_B^{p,q}$ the set (or sheaf of germs of) basic (p, q) -forms, and then we have natural operators

$$\partial_B : \Omega_B^{p,q} \rightarrow \Omega_B^{p+1,q}, \quad \bar{\partial}_B : \Omega_B^{p,q} \rightarrow \Omega_B^{p,q+1}.$$

They satisfy $\partial_B^2 = \bar{\partial}_B^2 = 0$. Let $H_B^{p,q}(N)$ be the corresponding cohomology groups which we call the basic cohomology group of type (p, q) . As we can define the Chern classes of compact complex manifolds using Hermitian metrics and its Ricci form, we can define Chern classes for transversely holomorphic foliations. They are Chern classes for the normal bundle $\nu(\mathcal{F}_\xi)$ of the Reeb flow \mathcal{F}_ξ . They can be expressed as basic cohomology classes, and for this reason they are called the *basic Chern classes*. In particular we have the first basic first Chern class

$$c_1^B(\nu(\mathcal{F}_\xi)) \in H_B^{1,1}(N).$$

Let $\omega^T := \frac{1}{2}d\eta$ be the transverse Kähler form. Suppose N is Sasaki-Einstein. Then since N is transversely Kähler-Einstein

$$(2m+2)[\omega^T] = c_1^B(\nu(\mathcal{F}_\xi)).$$

The following proposition gives a necessary condition for N to be Sasaki-Einstein.

Proposition 4.3. *Let N be a compact Sasaki manifold. Then there exists a real number κ with $c_1^B(\nu(\mathcal{F}_\xi)) = \kappa[\omega^T]$ for some $\kappa > 0$ if and only if $c_1(D) = 0$ and $c_1^B > 0$ where $D = \text{Ker } \eta$. Here $c_1^B > 0$ means that c_1^B is represented by a positive $(1, 1)$ -form on each local orbit space of the Reeb flow \mathcal{F}_ξ .*

Definition 4.4. *We say g is a transverse Kähler-Ricci soliton if*

$$\rho^T - \omega^T = L_X \omega^T$$

where ρ^T is the Ricci form of ω^T and X is a vector field on N which is obtained as the restriction to $N \cong \{r = 1\}$ of the real part of a holomorphic vector field \tilde{X} on $C(N)$ with $[\tilde{X}, \xi] = 0$.

It is well-known that, on a Fano manifold, if the Futaki invariant vanishes then Kähler-Ricci soliton is a Kähler-Einstein metric. In the similar way, we can define the Futaki invariant, which is called the Sasaki-Futaki invariant, and its various extensions for transverse Kähler structures on compact Sasaki manifolds, and in the case when $c_1(D) = 0$ and $c_1^B > 0$, if the Sasaki-Futaki invariant vanishes then any transverse Kähler-Ricci soliton is a transverse Kähler-Einstein metric. Therefore by Proposition 4.1 we obtain a Sasaki-Einstein metric if we can find a Sasaki manifold with $c_1(D) = 0$ and $c_1^B > 0$, with vanishing Sasaki-Futaki invariant and with a transverse Kähler-Ricci soliton.

Definition 4.5. *A Sasaki manifold N is toric if $C(N)$ is toric.*

Theorem 4.6 ([17]). *Let N be a compact toric Sasaki manifold with $c_1^B > 0$ and $c_1(D) = 0$. Then there exists a transverse Kähler-Ricci soliton.*

In the Fano manifold case, this theorem is due to X.-J. Wang and X.-H. Zhu [35].

By leaning the Reeb vector field ξ in the Lie algebra of the torus one can get another Reeb vector field with vanishing Sasaki-Futaki invariant. This is based on the idea by Martelli-Saprkis-Yau [31], called the volume minimization or Z-minimization in AdS-CFT correspondence. From this we get the following theorem.

Theorem 4.7 ([17]). *Let N be a compact toric Sasaki manifold with $c_1^B > 0$ and $c_1(D) = 0$. Then one can find a deformed Sasaki structure on which a Sasaki-Einstein metric exists.*

Now we wish to get a better understanding of the conditions for N to have $c_1^B > 0$ and $c_1(D) = 0$.

Theorem 4.8 ([8]). *Let N be a compact toric Sasaki manifold with $\dim N \geq 5$. We regard $C(N)$ as a toric variety including the apex. Then the following four conditions are equivalent.*

- (a) *N has $c_1^B > 0$ and $c_1(D) = 0$.*
- (b) *For some positive integer ℓ , the ℓ -th power $\mathcal{K}_{C(N)}^{\otimes \ell}$ of the canonical sheaf $\mathcal{K}_{C(N)}$ is trivial. In particular the apex is a \mathbb{Q} -Gorenstein singularity.*
- (c) *The Sasaki manifold N is obtained from a toric diagram with height ℓ for some positive integer ℓ .*

Next we wish to explain what the height is. For simplicity we put $n := m + 1$. A toric Kähler cone of complex dimension n is a Kähler cone having a Hamiltonian T^n -action. Its moment map image is a “good rational polyhedral cone”:

$$C = \{y \in \mathfrak{g}^* \mid \langle y, \lambda_i \rangle \geq 0, i = 1, \dots, d\}$$

where $G := T^n$, $\mathfrak{g} = \text{Lie}(G)$. There is an algebraic description of C due to Lerman [27], but we omit its detail and just define a good rational polyhedral cone as a moment cone obtained from a toric Kähler cone. As one of the algebraic properties we have

$$\lambda_i \in \mathbb{Z}_{\mathfrak{g}} := \text{Ker } \{\exp : \mathfrak{g} \rightarrow G\},$$

that is, every normal vector of facets is in the integral lattice of \mathfrak{g} .

Definition 4.9 (Toric diagram of height ℓ). *We say $\{\lambda_i\}_{i=1}^d \subset \mathbb{Z}^n \cong \mathbb{Z}_{\mathfrak{g}}$ and $\gamma \in \mathbb{Q}^n \cong (\mathbb{Q}_{\mathfrak{g}})^*$ define a toric diagram of height ℓ if*

- (1) $C = \{y \in \mathfrak{g}^* \mid \langle y, \lambda_i \rangle \geq 0, i = 1, \dots, d\}$ is a good rational polyhedral cone (i.e. the moment map image of a toric Kähler cone).
- (2) ℓ is the largest integer such that $\ell\gamma$ is in \mathbb{Z}^n and primitive.
- (3) $\langle \gamma, \lambda_i \rangle = -1$.

Using an element of $SL(m+1, \mathbb{Z})$ we can transform γ and λ_j 's so that

$$\gamma = \begin{pmatrix} -\frac{1}{\ell} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and all λ_j are of the form

$$\lambda_j = \begin{pmatrix} \ell \\ \vdots \\ \vdots \end{pmatrix}.$$

This is the reason why we call “height ℓ ”.

The height is related to the fundamental group of N . Let \mathcal{L} be the subgroup of $\mathbb{Z}_{\mathfrak{g}}$ generated by $\lambda_1, \dots, \lambda_d$. Then by Lerman [28] we have

$$\pi_1(N) \cong \mathbb{Z}_{\mathfrak{g}} / \mathcal{L}.$$

Note that $\mathbb{Z}_{\mathfrak{g}} / \mathcal{L}$ is not trivial if $\ell > 1$. Thus if N is a compact connected toric Sasaki manifold associated with a toric diagram of height $\ell > 1$, then N is not simply connected. The converse is not true as the following example shows. If we take

$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

then $\pi_1(N) = \mathbb{Z}_5$.

5. THE CONSTRUCTION OF SPECIAL LAGRANGIAN SUBMANIFOLDS AND LAGRANGIAN SELF-SHRINKERS IN TORIC CALABI-YAU CONES.

Recall from the previous section that a Sasaki manifold (N, g) of dimension $2m+1$ is a toric Sasaki-Einstein manifold if the cone $(C(N), \bar{g})$ is a toric Calabi-Yau cone, i.e. Ricci-flat toric Kähler cone of complex dimension $m+1 =: n$. The apex of toric Kähler cone $C(N)$ is \mathbb{Q} -Gorenstein singularity if and only if the moment cone comes from a toric diagram of height ℓ . This is equivalent to say that $K_{C(N)}^{\otimes \ell}$ is trivial. In particular $C(N)$ is Calabi-Yau if $\ell = 1$. From now on, we shall have toric diagrams of *height* 1 in mind even if we use general ℓ . By combining Theorem 4.7 and Theorem 4.8 we obtain the following.

Theorem 5.1 ([17], [8]). *For every toric cone manifold $C(N)$ coming from toric diagram of height ℓ , there exists a Ricci-flat Kähler cone metric. (Equivalently, N admits a Sasaki-Einstein metric.)*

Suppose $\ell = 1$. Then one can show that

$$\Omega = e^{-\sum \gamma_i z^i} dz^1 \wedge \cdots \wedge dz^n$$

is a parallel holomorphic n -form, see [8] for the proof. This implies that

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \binom{\sqrt{-1}}{2}^n \Omega \wedge \overline{\Omega}$$

for some Kähler form ω . If

$$\gamma = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

then we have

$$\Omega = e^{z^1} dz^1 \wedge \cdots \wedge dz^n.$$

The following definitions and theorems 5.2 – 5.5 are due to Harvey-Lawson [21], and now are standard in differential geometry.

Definition 5.2. *Let (M, g) be a Riemannian manifold. A closed k -form Ω on (M, g) is a calibration if for any $p \in M$ and for any k -dimensional subspace $V \subset T_p M$,*

$$\Omega|_V \leq \text{dvol}_V(g).$$

If a k -submanifold L satisfies $\Omega|_L = \text{dvol}_L(g)$, L is called a calibrated submanifold.

Theorem 5.3. *A closed calibrated submanifold minimizes volume in its homology class.*

Theorem 5.4. *Let (M, Ω, g) be a Calabi-Yau manifold. Then $\text{Re}\Omega$ defines a calibration.*

Definition 5.5. *A calibrated submanifold L for $\text{Re}\Omega$ on a Calabi-Yau manifold (M, Ω, g) is called a special Lagrangian submanifold:*

$$\dim L = n, \omega|_L = 0, \text{ and } \text{Im}\Omega|_L = 0.$$

Theorem 5.6 ([15]). *With $\ell = 1$, there is a T^{n-1} -invariant special Lagrangian submanifold L described as*

$$L = \mu^{-1}(c) \cap \{(e^{z^1} + (-1)^n e^{\overline{z^1}})/i^n = c'\}$$

where T^{n-1} is generated by $\text{Im}(\partial/\partial z^2), \dots, \text{Im}(\partial/\partial z^n)$ and $\mu : C(N) \rightarrow \mathfrak{t}^{n-1}$ the moment map.*

Example 5.7. *If $N = S^{2n-1}$ then we have $C(N) = \mathbb{C}^n - \{\mathbf{o}\}$. Let w^1, \dots, w^n be the holomorphic coordinates on \mathbb{C}^n . In this case, the conditions in Theorem 5.6 are equivalent to*

- (1) $|w^j|^2 - |w^1|^2 = c_j, j = 2, \dots, n.$
- (2) *If n is even, $\text{Re}(w^1 \cdots w^n) = c'.$
If n is odd, $\text{Im}(w^1 \cdots w^n) = c'.$*

This special Lagrangian submanifold is due to Harvey-Lawson [21].

See also Kawai [25] for a similar construction.

Theorem 5.8 (Yamamoto [36]). *Let g be any non-negative integer and Σ_g be a compact surface of genus g . Then there exists a toric Calabi-Yau cone $C(N)$ of height 1 with $\dim_{\mathbb{C}} C(N) = 3$ such that in $C(N)$*

- (1) *there exist special Lagrangian submanifolds diffeomorphic to $\Sigma_g \times \mathbb{R}$ and*
- (2) *there exist Lagrangian self-shrinkers diffeomorphic to $\Sigma_g \times S^1$.*

6. THE DIAMETER OF COMPACT SELF-SHRINKERS FOR MEAN CURVATURE FLOW

Let $x : M \rightarrow \mathbb{R}^{n+p}$ be an n -dimensional submanifold in the $(n+p)$ -dimensional Euclidean space. If we let the position vector x evolve in the direction of the mean curvature \vec{H} , then it gives rise to a solution to the mean curvature flow :

$$x : M \times [0, T) \rightarrow \mathbb{R}^{n+p}, \quad \frac{\partial x}{\partial t} = \vec{H}.$$

We call the immersed manifold M a self-shrinker if it satisfies the quasilinear elliptic system (see [22], or [9]): for some positive constant λ ,

$$\vec{H} = -\lambda x^\perp,$$

where \perp denotes the projection onto the normal bundle of M .

We have (see [29])

$$\frac{1}{2\lambda}|\vec{H}|^2 + \frac{1}{4}\Delta|x|^2 = \frac{n}{2}.$$

Put

$$\phi := 2\lambda\left(\frac{|x|^2}{4} - \frac{n}{4\lambda}\right). \quad (15)$$

Define the Witten-Laplacian by

$$\Delta_\phi = \Delta - \nabla\phi \cdot \nabla.$$

From above formulas, we can check

$$\begin{aligned} \Delta_\phi\left(\frac{1}{4}|x|^2\right) &= \Delta\left(\frac{1}{4}|x|^2\right) - \frac{\lambda}{8}\nabla|x|^2 \cdot \nabla|x|^2 \\ &= \frac{n}{2} - \frac{1}{2\lambda}|\vec{H}|^2 - \frac{\lambda}{2}|x^T|^2 \\ &= \frac{n}{2} - \frac{\lambda}{2}|x|^2. \end{aligned}$$

Thus we have

$$\Delta_\phi\left(\frac{1}{4}|x|^2 - \frac{n}{4\lambda}\right) = -2\lambda\left(\frac{|x|^2}{4} - \frac{n}{4\lambda}\right).$$

Thus we have proved

Theorem 6.1 ([16]). *In the above situation we have the eigenvalue 2λ of the Witten-Laplacian Δ_ϕ with eigenfunction ϕ :*

$$\Delta_\phi\phi = -2\lambda\phi.$$

Thus we have obtained an eigenvalue similar to (2), and can hope to get a diameter estimate similar to the case of Ricci solitons. In fact we can show the following theorem.

Theorem 6.2 ([16]). *Let $x : M \rightarrow R^{n+p}$ be an n -dimensional compact self-shrinker such that $x(M)$ is not minimal submanifold in $S^{n+p-1}(\sqrt{n/\lambda})$, and let h_{ij}^α be the components of the the second fundamental form of M . Then we have*

$$d \geq \frac{1}{\sqrt{\frac{3\lambda}{2} + \frac{1}{2}K_0}}\pi,$$

where

$$K_0 := \max_{1 \leq i \leq n} \left[\sum_{\alpha, k} h_{ik}^\alpha h_{ki}^\alpha \right].$$

When $p = 1$, we have

Corollary 6.3 ([16]). *Let $x : M \rightarrow R^{n+1}$ be an n -dimensional compact self-shrinker such that $x(M)$ is not $S^n(\sqrt{n/\lambda})$, and let λ_i be the principal curvatures of M . Then we have*

$$d \geq \frac{1}{\sqrt{\frac{3\lambda}{2} + \frac{1}{2}K_0}}\pi,$$

where

$$K_0 := \max_{p \in M} \max_{1 \leq i \leq n} \lambda_i^2.$$

7. ETERNAL SOLUTIONS TO KÄHLER-RICCI FLOW

Let M be a toric Fano manifold of $\dim M = m$, and $L \rightarrow M$ be a line bundle over M with $K_M = L^{-p}$, $p \in \mathbb{Z}_+$. The claim of this section is that we can construct Kähler-Ricci solitons on L^{-k} outside the zero section using Calabi ansatz starting with Sasaki-Einstein metrics on the associated $U(1)$ -bundle of L^{-k} .

A Kähler-Ricci flow is a family ω_t of Kähler forms satisfying

$$(9) \quad \frac{d}{dt}\omega_t = -\rho(\omega_t)$$

where $\rho(\omega)$ is the Ricci form of ω . A Kähler-Ricci soliton is a Kähler form ω satisfying

$$(10) \quad -\rho(\omega) = \lambda\omega + \mathcal{L}_X\omega$$

for some holomorphic vector field X

where $\lambda = 1, 0$ or -1 . When

$$\mathcal{L}_X\omega = i\partial\bar{\partial}u$$

for some real function u , we say that the Kähler-Ricci soliton is a gradient Kähler-Ricci soliton. According as $\lambda = 1, 0$ or -1 the soliton is said to be expanding, steady and shrinking.

Given a Kähler-Ricci soliton with $\lambda = \pm 1$, let γ_t be the flow generated by the time dependent vector field

$$Y_t := \frac{1}{\lambda t}X.$$

Then

$$\omega_t := \lambda t \gamma_t^* \omega$$

is a Kähler-Ricci flow. Notice that, when $\lambda = 1$, the Ricci flow exists for $t > 0$ and that, when $\lambda = -1$, the Ricci flow exists for $t < 0$. When $\lambda = 0$ if we put

$$\omega_t := \gamma_t^* \omega$$

where γ_t is the flow generated by the vector field X , then ω_t is a Kähler-Ricci flow.

Theorem 7.1 ([19]). *Suppose $0 < k < p$.*

- (1) *There exists a shrinking soliton on $L^{-k} - \{\text{zero section}\}$. (The flow exists for $t < 0$.)*
- (2) *There exists an expanding soliton on $L^{-k} - \{\text{zero section}\}$. (The Ricci flow exists for $t > 0$.)*
- (3) *The first one and the second one can be pasted to form an eternal solution of the Ricci flow on $(L^{-k} - \{\text{zero section}\}) \times (-\infty, \infty)$.*

Remark 7.2. *For $p < k$, there exists an expanding soliton on $(L^{-k} - \{\text{zero section}\})$.*

For $M = \mathbb{CP}^m$, $L^{-1} = \mathcal{O}(-1)$, (2) is due to H.-D. Cao, and extends to \mathbb{C}^{m+1} , and (1) and Remark 7.2 are due to Feldman, Ilmanen and Knopf [12]. The solutions in (1) and Remark 7.2 due to [12] extend to the zero section. So, the solution exists on

$$L^{-1} \times (-\infty, 0) \cup (\mathbb{C}^{m+1} - \{\mathbf{o}\}) \times \{0\} \cup \mathbb{C}^{m+1} \times (0, \infty).$$

Now we explain the Calabi ansatz (momentum construction) in the following classical case. Let (M, ω) be a Fano Kähler-Einstein manifold, and search for a Ricci-flat Kähler metric on K_M .

Let $p : K_M \rightarrow M$ be the canonical line bundle, h an Hermitian metric of K_M such that

$$i\partial\bar{\partial} \log h = \omega.$$

Define $r : K_M \rightarrow \mathbb{R}$ by $r(z) = \sqrt{h(z, z)}$. Search for a Kähler metric of constant scalar curvature of the form $\tilde{\omega} = p^*\omega + i\partial\bar{\partial}f(r)$ where $f(r)$ is a smooth function of r . We obtain a 2nd order ODE in terms of $\varphi(\sigma) := f''(r)$ where $\sigma = f'(r)$. Although the equation was set up to find a constant scalar curvature Kähler metric, the metric obtained happens to be Ricci-flat. Hence we have obtained a Ricci-flat Kähler metric except the zero section. Next task is to find conditions so that the metric extends smoothly to the zero section. The answer is the following:

$$\varphi(0) = 0 \quad \text{and} \quad \varphi'(0) = 2.$$

Another task is to find conditions so that the metric becomes complete near $r = \infty$. The answer in this case is the following:

$$\varphi(r) = O(r^2) \quad \text{as} \quad r \rightarrow \infty.$$

Instead of Kähler-Einstein manifolds, we use Sasaki-Einstein manifolds. The same idea applies for the construction of gradient Kähler-Ricci solitons except the extension to the zero section. The difficulty of the extension to the zero section arises when the Reeb vector field is irregular. As a version of Theorem 5.1 we have the following. Suppose

- (i) Suppose that M is toric Fano, and take the Kähler class $[\omega] = c_1(M)$,
- (ii) consider the Sasaki manifold $N \subset K_M$ to be the associated $U(1)$ -bundle.

Then N admits a Sasaki-Einstein metric possibly with irregular Reeb vector field.

The Calabi ansatz for Sasakian manifold N is described as

$$p : C(N) = K_M - \{\text{zero section}\} \rightarrow N,$$

$$\tilde{\omega} = p^* \left(\frac{1}{2} d\eta \right) + i\partial\bar{\partial}f(r).$$

Summary of this section is :

1. If M is a toric Fano manifold then $U(1)$ -bundle N associated with K_M is a toric Sasakian manifold.
2. By Futaki-Ono-Wang [17] N admits a possibly irregular Sasaki-Einstein metric.
3. We can apply Calabi's ansatz to get Kähler-Ricci solitons in L^k outside the zero section for L with $L^{-p} = K_M$ and $0 < k < p$.

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